Pre-class Warm-up!!!

Which of the following systems of equations is equivalent to the 2nd order equation
x″ - 3x′ + 2x = 0 ?

a. $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

c. $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

d. None of the above.

$y = x' \quad y' = 3x' - 2x = 3y - 2x$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} y \\ 3y - 2x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

We have the characteristic polynomial
$r^2 - 3r + 2$ (of the d.e.)

and the characteristic polynomial of $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ $\det\begin{bmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix}$

$= -\lambda(3-\lambda) + 2 = \lambda^2 - 3\lambda + 2$.

They are the same!

Section 7.2 Matrices and linear systems

We learn about:

- writing a linear system of equations in vector form
- several theorems similar to ones for higher order d.e.'s we have already seen
- the Wronskian again.

Write $X' = PX + F$

where $X(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}$

$F = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$

instead of $x_1' = p_{11}x_1 + p_{12}x_2 + \cdots + f_1$

$\vdots$

$x_n' =$

A system of equations $X' = PX + F$ is homogeneous if $F = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{0} = 0$

Taking derivatives is a linear operator: If X_1, X_2 are vector valued functions, c_1, c_2 are scalars then

$$(c_1 X_1 + c_2 X_2)' = c_1 X_1' + c_2 X_2'$$
$$= P(c_1 X_1 + c_2 X_2) = c_1 P X_1 + c_2 P X_2$$

The principle of superposition of solutions:

If X_1, X_2 are solutions to a homogeneous system then so is $c_1 X_1 + c_2 X_2$

Theorem 1 The solutions to a homogeneous system form a vector space.

Theorem 3 The space has dimension n if P and F are continuous.

The Wronskian of vector valued functions X_1, … ,X_n is

$$\det\left[X_1 \middle| \cdots \middle| X_n\right] = W(t)$$

matrix whose columns are $X_1 \cdots X_n$.

Theorem 2
(a) If X_1, … ,X_n are dependent then W = 0.
(b) If they are also solutions of a homogeneous linear system and they are independent, then W is never 0.

Proof (a) If they are dependent then there is a nonzero dependence relation between the columns of $[X_1|\cdots|X_n]$ (one is a linear combination of some others). This means the det. is 0.

The connection with the Wronskian of scalar-valued functions f_1, … ,f_n.

That Wronskian was

$$\det\begin{bmatrix} f_1 & \cdots & f_n \\ f_1' & & \\ \vdots & & \\ f_1^{(n-1)} & & f_n^{(n-1)} \end{bmatrix}$$

In converting from a high order d.e. in one variable we introduced variables $x_1 = x$, $x_2 = x_1'$, $x_3 = x_2'$ …

producing vectors $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x \\ x' \\ x'' \\ \vdots \\ x^{(n-1)} \end{bmatrix}$

The two Wronskians are the same when we convert a high order d.e. in one variable to a first order system in several variables.

Page 384 question 14.
Verify that the given vectors are solutions of the differential equation. Use the Wronskian to show that they are independent.

$$\underbrace{X' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} X}_{\text{d.e.}}, \quad \underbrace{X_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}, \; X_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}}_{\text{solutions}}$$

Solution: Check X_1 is a solution!

$$\begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} (-3+6)e^{3t} \\ (-3+12)e^{3t} \end{bmatrix} \checkmark$$

$$W(t) = \det \begin{bmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{bmatrix}$$

$$= e^{t} - 6e^{t} = -5e^{t}$$

This is not the zero function. In fact it is never 0.

Thus X_1, X_2 are independent.

Page 384 question 23.
Find a particular solution of the system in question 14 that satisfies x_1(0) = 0, x_2(0) = 5

component 1 of the solution cmpt 2

Solution: We look for a solution

$$A\begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} + B\begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}. \quad \text{Put } t = 0$$

$$A\begin{bmatrix} 1 \\ 3 \end{bmatrix} + B\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -10 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The particular solution is

$$\begin{bmatrix} 2e^{3t} - 2e^{-2t} \\ 6e^{3t} - e^{-2t} \end{bmatrix}$$

Question.

What is the Wronskian of the functions

$$\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}, \quad \begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix} ?$$

a. 1

b. -1

c. 0

d. $\cos^2 2t - \sin^2 2t$

e. None of the above.

Theorem 1 of section 7.1.
In the first order linear system $X' = PX + F$ if the functions P and F are continuous then, given numbers $a, b_1, \ldots, b_n$, there is a unique solution satisfying
$x_1(a) = b_1, x_2(a) = b_2, \ldots, x_n(a) = b_n$.

We conclude:

Theorem 3.
The space of solutions of a homogeneous first order linear system in n variables has dimension n.

Deduction of this: Take a basis $X_1, \ldots, X_d$ for the space of solutions.

For each $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ there is a unique linear combination

$$c_1 X_1(a) + \cdots + c_d X_d(a) = B.$$

$$[X_1 | \cdots | X_d](a) \begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix} = B$$

has a unique unique solution in $(c_1, \ldots, c_d)$.

It follows that $[X_1 | \cdots | X_d]$ is square, so $d = n$.